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Time-domain Green's functions for unsaturated soils. Part I: Two-dimensional solution

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Abstract

In this article, primarily a brief discussion about the formulation of unsaturated soils including the equilibrium, air and moisture transfer equations is presented. Then the closed form *two-dimensional* Green's functions of the governing differential equations for an unsaturated deformable porous medium with linear elastic behavior for a symmetric polar domain in both Laplace transform and time domains have been introduced, for the first time. Using the linear form of the governing differential equations and considering the effects of non-linearity of the governing equations, the Green's functions have been derived exactly and verified in both Laplace transform and time domains.

Keywords: Green's function; Unsaturated soil; Laplace transform; Boundary element method

1. Introduction

Prediction and simulation of unsaturated soil behavior are of great importance in making critical decisions that affect many facets of engineering design and construction and, therefore, have been the issue of growing concern for several decades. In order to model unsaturated soil behavior, firstly the governing partial differential equations should be derived and solved. Regarding the form and combination of the governing partial differential equations, with the exception of some simple cases, the closed form solutions of the partial differential equations are not available. Therefore the numerical techniques have been widely used for such partial differential equations. Both finite and boundary element methods (BEM) have been

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used for obtaining the response of such domains. The finite element method, regarding its vast ability in geomechanics as well as many other areas, has been used in many codes that have been developed for both saturated and unsaturated cases, though the necessity of finding the Green's functions for the governing partial differential equations to develop a BEM model seems to have reduced the development rate of BEM in different fields.

Many researches have been focused on deriving the fundamental solutions of the governing partial differential equations for saturated media that have successfully resulted in developing some BEM models for saturated soils (Nowacki, 1966; Cleary, 1977; Burridge and Vargas, 1979; Cheng and Liggett, 1984a,b; Norris, 1985; Kaynia and Banerjee, 1992).

According to the authors' knowledge, the fundamental solutions of the governing partial differential equations for unsaturated porous media have not been developed so far, hence the development of a BEM model for unsaturated phenomenon is not yet possible. The present research is an effort to derive such Green's functions in order to develop a BEM formulation and model for unsaturated soils.

2. Literature review

There are numerous media encountered in engineering practice whose behavior is not consistent with the principles and concepts of classical saturated soil mechanics. Commonly it is the presence of more than two phases that results in a media that is difficult to deal with in engineering applications. Unsaturated soils form the largest category of materials, that cannot be classified by classical saturated soil mechanics concepts. Any soil near the ground surface in a relatively dry environment will be subjected to negative pore-water pressure and possible desaturation.

An unsaturated soil is commonly characterized by three phases, soil solids, water, and air. Although there is a debate over existence of a fourth phase, a so called air–water interface or contractile skin (Fredlund and Morgenstern, 1977), in this research three phases approach is adopted.

The first ISSMFE conference held in 1936 provided a forum for the establishment of principles and equations relevant to saturated soil mechanics. Researches at Imperial College began to establish the basic concepts of unsaturated soils behavior in the late 1950s (Bishop, 1959).

One of the first problems that appeared to confuse civil engineers was the movement of water above the ground water table. The term 'capillary' was adopted to describe the phenomenon of water flow upward from the static ground water table. Hogentogler and Barber (1941) attempted to present a comprehensive review of the nature of the capillary.

Terzaghi (1943) in his book 'Theoretical soil mechanics' summarized the mentioned researches and endorsed the concepts related to the capillary tube model. He derived an equation for the time required for the rise of water in capillary zone, that appears to overestimate the rate of capillary rise. Lambe (1951) performed the open tube capillary rise and drainage tests on graded sands and silts with various initial degrees of saturation.

The mechanical behavior of a soil can be described in terms of the state of stress in the soil. The state of stress in soil consists of certain combinations of stress variables that can be referred to as stress state variables. The number of stress variables required for the description of the stress state of a soil depends primarily upon the number of phases involved. The effective stress is simply a stress state variable that can be used to describe the behavior of a saturated soil. The volume change process and the shear strength characteristics of a saturated soil are both controlled by a change in the effective stress.

In 1941 Biot proposed a general theory of consolidation for an unsaturated soil with occluded air bubbles. The constitutive equations relating stress and strain were formulated in terms of the effective stress ($\sigma - p_w$) and the pore water pressure p_w (Biot, 1956a). In the other words, the need for separating the effects of total stress and pore-water pressure was recognized.

Bishop (1959) suggested a tentative expression for effective stress that has gained widespread reference

$$\sigma' = (\sigma - p_a) + \chi(p_a - p_w) \quad (1)$$

in which σ and p_a stand for stress and air pressure, respectively. The magnitude of χ parameter is unity for a saturated soil and zero for a dry soil. The relationship between χ and the degree of saturation, S_r, was obtained experimentally.

Jennings and Burland (1962) appear to be the first to suggest that Bishop's equation did not provide an adequate relationship between volume change and effective stress for most soils, particularly those below a critical degree of saturation. That was estimated to be approximately twenty percents for silts and sands and as high as ninety percents for clays.

Coleman (1962) suggested the use of 'reduced' stress variables $(\sigma_1 - p_a)$, $(\sigma_3 - p_a)$ and $(p_w - p_a)$ to represent the axial confining and pore-water pressures, respectively, in triaxial tests.

In 1963 Bishop and Blight re-evaluated the proposed effective stress equation for unsaturated soil. It was noted that a variation in matric suction $(p_a - p_w)$ did not result in the same change in effective stress as did change in the net normal stress $(\sigma - p_a)$.

Aitchison (1967) pointed out the complexity associated with the χ parameter. He stated that a specific value of χ may only relate to a single combination of σ and $(p_a - p_w)$ for particular stress path.

Matyas and Radhakrishna (1968) introduced the concept of 'state parameters' in describing the volumetric behavior of unsaturated soils. Volume change was presented as a three-dimensional surface with respect to the state parameters $(\sigma - p_a)$ and $(p_a - p_w)$. Barden et al. (1969) also suggested that the volume change of unsaturated soils could be analyzed in terms of the separate components of applied stress, $(\sigma - p_a)$, and suction, $(p_a - p_w)$.

Numerous effective stress equations have been proposed incorporating a soil parameter in order to form a single valued effective stress variable, but experiments have demonstrated that the effective stress equation was not single valued and there was a dependence on the stress path followed. Re-examination of the proposed effective stress equations had led many researchers to suggest the use of independent stress variables $(\sigma - p_a)$ and $(p_a - p_w)$ to describe the mechanical behavior of unsaturated soils.

Fredlund and Morgenstern (1976–1977) presented a theoretical stress analysis of an unsaturated soil, based on multiphase continuum mechanics. They concluded that any two of three possible normal stress variables can be used to describe the stress state of an unsaturated soil. In other words, there are three possible combinations that can be used as stress state variables for an unsaturated soil. The stress state variables can be used to formulate constitutive equations to describe the shear strength and the volume change behavior of unsaturated soils.

Historically, classical mathematics was the main tool for solving governing differential equations of various problems in engineering practice. With the advent of high-speed digital computers, increasing number of engineering analyses are performed via computational methods such as finite differences, finite elements and boundary elements methods vastly in use since the 1960s. However it needs to be emphasized though appearing trivial and repetitive, that computational methods can, and in many cases should, benefit from classical mathematical analysis. This is especially true in the case of BEM where a specific and important subject is to determine the fundamental solutions and boundary integral equations pertaining to governing differential equations via classical mathematics.

The corresponding fundamental solutions for governing differential equations of saturated soils have been introduced through the last decades. Cleary (1977) derived the fundamental solutions for quasi-static problem following the earlier work of Nowacki (1966). Closed form Laplace transform domain quasi-static poroelastic fundamental solutions were obtained by Cheng and Liggett (1984a,b). The first attempt to obtain fundamental solutions for dynamic poroelasticity seems to be presented by Burridge and Vargas (1979) who presented a general solution procedure similar to that of Deresiewicz (1960). Later, Norris (1985) derived time harmonic Green's functions for a point force in the solid and a point force in the fluid.

Afterwards, Kaynia and Banerjee (1992) used a solution scheme similar to that of Norris (1985) and derived the fundamental solution in the Laplace transform domain as well as transient short-time solution.

The Burridge and Vargas solution was obtained for three forces, while those of Norris and Kaynia and Banerjee have used six variables (displacements of the solid skeleton and average displacements of the fluid), both of which seem to be completely inadequate. The first approach does not have enough variables and the second one has too much.

The well-known time harmonic poroelastic fundamental solutions were introduced by Bonnet (1987) and Boutin et al. (1987); but they are not without drawback either. The errors in Bonnet's paper have been pointed out by Dominguez (1991, 1992). Additionally, Bonnet's solution dos not allow clear identification of the sources involved in the calculation, as was noted by Boutin et al. (1987). Boutin on the other hand worked on the equations that are based upon the homogenization theory for periodic structures (Auriault, 1980; Auriault et al., 1985). However Boutin's solution is in symmetrical form, while the Green's functions for this problem should not be symmetric. Also Weibe and Antes (1991) seem to be the first which obtained a time domain fundamental solution for the Biot (1956a,b,c) type dynamic poroelasticity by neglecting the viscous coupling and without numerical evaluation of the kernel functions.

Finally, Chen (1994a,b) provided analytical time domain Green's functions for two- and three-dimensional full dynamic poroelasticity in two separate papers. Thereupon, Gatmiri and Kamalian (2002) have modified Chen's two-dimensional solution and boundary integral formulation to lead to more accurate results. Also Gatmiri and Nguyen (2005) have derived closed form Green's functions for two-dimensional saturated soil with incompressible fluid. They have shown that their solution is a good approximation of the exact solution, especially for the long time.

More recently Schanz and Pryl (2004) have derived dynamic fundamental solutions for deformable soil's solid skeleton with compressible and incompressible fluid in Laplace transform domain. By comparison of the two sets of the derived Green's functions they have concluded that an incompressible model can only be used in wave propagation problems if not the short time behavior is considered and also if the ratios of the compression moduli are very insignificant.

The present research is an effort for deriving these Green's functions for *two-dimensional* deformable quasi-static unsaturated soil. Following some reasonable and necessary simplifications, the fundamental solutions will be introduced in both frequency and time domains, for the first time. Although, two- and three-dimensional poroelastostatic Green's functions for unsaturated soils have been introduced by the authors (Gatmiri and Jabbari, 2004a,b) for time-independent problems.

3. Governing equations

In unsaturated porous media (Fig. 1) the governing differential equations consist of equilibrium equations, constitutive equations of the solid skeleton, and continuity and transfer equations for air and water. These equations may be written as follow (Gatmiri et al., 1998).

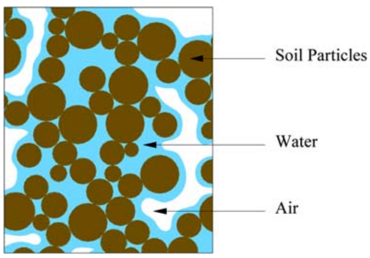

Fig. 1. Unsaturated soil scheme.

3.1. Equilibrium and constitutive equations of solid skeleton

Equilibrium equations based on the two independent parameters $(\sigma - p_a)$ and $(p_a - p_w)$, with elastic or linear behavior are

$$(\sigma_{ij} - \delta_{ij}p_a)_{,j} + p_{a,i} + b_i = 0 \tag{2}$$

and stress–strain relations

$$d(\sigma_{ij} - \delta_{ij}p_a) = D_{ijkl}d\varepsilon_{kl} + \delta_{ij}D_s(dp_a - dp_w) \tag{3}$$

or

$$(\sigma_{ij} - \delta_{ij}p_a) = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} + \delta_{ij}D_s(p_a - p_w). \tag{4}$$

Considering the strain-deformation relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

the final equation stating the equilibrium of solid skeleton becomes

$$(\lambda + \mu)u_{j,ij} + \mu u_{i,jj} + (D_s - 1)p_{a,i} - D_s p_{w,i} + b_i = 0. \tag{6}$$

In Eqs. (2)–(6) λ and μ are Lamé's coefficients, D_{ijkl} are the coefficients of soil elasticity and D_s is the coefficient of deformations due to suction effect. In addition, σ, ε, u and b stand for stress, strain, displacement of soil's solid skeleton and the body forces, respectively. Also, p_a and p_w denote air and water pressures and δ_{ij} represents the Kronecker delta operator.

3.2. Continuity and transfer equations for air

A combination of generalized Darcy's (1856) law for air transfer and conservation law for air mass, leads to the general equation for air transfer. The air velocity, u_a, is defined as

$$u_a = -K_a\nabla\left(\frac{p_a}{\gamma_a} + z\right), \tag{7}$$

where γ_a and z are air unit weight and the element's height from an arbitrary level, respectively. The air coefficient of permeability, K_a, is defined as

$$K_a = D_K\frac{\gamma_a}{\mu_a}[e(1 - S_r)]^{E_K}, \tag{8}$$

where μ_a, e and S_r are air dynamic viscosity, void ratio and degree of saturation, respectively and D_K and E_K are constants (Lambe and Whitman, 1969).

In the similar manner, the water velocity, u_w, is

$$u_w = -K_w\nabla\left(\frac{p_w}{\gamma_w} + z\right) \tag{9}$$

in which γ_w is water unit weight. K_w is the water permeability and is defined as (Kovacs, 1981)

$$K_w = K_{wz0}\left(\frac{S_r - S_{ru}}{1 - S_{ru}}\right)^{3.5}, \tag{10}$$

where S_{ru} is residual degree of saturation and K_{wz0} is the intrinsic water permeability defined as

$$K_{wz0} = a_{K_w}10^{\alpha_{K_w}e}, \tag{11}$$

where a_{Kw} and α_{Kw} are constant coefficients.

The mass conservation law for air unit volume is written as (Alonso et al., 1988)

$$\frac{\partial}{\partial t}\{\rho_a n[1 - S_r(1 - H)]\} + \mathrm{div}[\rho_a(u_a + Hu_w)] = 0 \tag{12}$$

in which H is the Henry's coefficient and denotes the amount of dissolved air in water, ρ_a is air density, t is time variable and n stands for porosity.

Assuming constant ρ_a and K_a in space, dispensing with variations of ρ_a in time, and remembering that the Laplacian of z is zero, we have

$$\frac{\rho_a K_a}{\gamma_a}\nabla^2 p_a + \frac{H\rho_a K_w}{\gamma_w}\nabla^2 p_w = \rho_a \frac{\partial}{\partial t}[n(1 - S_r(1 - H))], \tag{13}$$

where div, ∇ and ∇^2 stand for divergence, gradient and Laplacian operators.

We note that according to Eq. (8) K_a is a function of S_r which corresponds to physical reality of air flow in unsaturated media. Keeping K_a as a function dependent of S_r and consequently of $(p_a - p_w)$ makes the differential equation non-linear (or with variable coefficients) so that deriving the considered Green's functions will become too difficult, at least with common methods. Therefore as a first step of deriving the Green's functions, it is reasonable to keep the effects of K_a by using S_r values in different constant suction areas. Consequently, the effects of S_r have been considered in air coefficient of permeability and the basic physical concept of the effects of S_r is preserved by assuming K_a as a step function of $(p_a - p_w)$ for each area.

One can write the right-hand side of Eq. (13) as

$$\begin{aligned}\rho_a \frac{\partial}{\partial t}[n(1 - S_r(1 - H))] &= \rho_a\left[(1 - S_r(1 - H))\frac{\partial}{\partial t}(n) - n(1 - H)\frac{\partial}{\partial t}(S_r)\right] \\ &= \rho_a\left[(1 - \widehat{S}_r(1 - H))\frac{\partial}{\partial t}(n) - \hat{n}(1 - H)\frac{\partial}{\partial t}(S_r)\right],\end{aligned} \tag{14}$$

where the hat sign ($^\wedge$) denotes that the parameter is assumed constant during the infinitesimal period ∂t. The porosity, n, may be written as

$$n = \varepsilon_v = \varepsilon_{ii} = u_{i,i}. \tag{15}$$

Numerous relations have been introduced to define the degree of saturation of unsaturated soils, but the logarithmic form based on suction variations is one of the most common and reliable ones. Logarithmic form of the degree of saturation is chosen here in the form of (Fredlund and Rahardjo, 1993):

$$S_r = \alpha + \beta \log(p_a - p_w), \tag{16}$$

where α and β are constants. By choosing $(\alpha = 1)$ and assuming a negative β, one can see that any increase in suction results a decrease in S_r and any decrease in suction results the approach of S_r to one (saturated).

Considering the Taylor expansion of $\ln(x)$ about $x = 0$ (Spiegel, 1999)

$$\ln(1 + x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{17}$$

and keeping the first term [1], we have

$$S_r = \alpha + \beta(p_a - p_w) \tag{18}$$

[1] Keeping more than the first power of x will make the governing differential equations too complicated such that we will not be able to apply a Laplace transform. In addition, many references use the equation, especially in small values of stress and suction, in the linear form.

consequently, one may write the right-hand side of Eq. (13) as

$$\rho_a[1-(\alpha+\beta(\hat{p}_a-\hat{p}_w))(1-H)]\frac{\partial}{\partial t}(u_{i,i})-\rho_a\beta\hat{u}_{i,i}(1-H)\frac{\partial}{\partial t}(p_a-p_w) \tag{19}$$

and finally the air transfer equation or Eqs. (12) and (13) will be

$$\begin{aligned}&\frac{\rho_a K_a}{\gamma_a}\nabla^2 p_a+\frac{H\rho_a K_w}{\gamma_w}\nabla^2 p_w\\&\quad=-\rho_a\beta\hat{u}_{i,i}(1-H)\frac{\partial}{\partial t}(p_a-p_w)+\rho_a[1-(\alpha+\beta(\hat{p}_a-\hat{p}_w))(1-H)]\frac{\partial}{\partial t}(u_{i,i}).\end{aligned} \tag{20}$$

3.3. Continuity and transfer equations for water

Again by applying the generalized Darcy's (1856) law for water transfer and mass conservation law together, one can obtain the same relation for water (Alonso et al., 1988).

Applying the mass conservation law for water, the water transfer equation will be

$$\frac{\partial}{\partial t}[\rho_w n S_r]+\mathrm{div}[\rho_w u_w]=0, \tag{21}$$

where ρ_w is water density.

Considering Eq. (9) and again assuming constant ρ_w and K_w in space and dispensing with variations of ρ_w in time, we have

$$\frac{\rho_w K_w}{\gamma_w}\nabla^2 p_w=\rho_w\frac{\partial}{\partial t}[nS_r]. \tag{22}$$

A discussion similar to that made for K_a shows that it is inevitable to dispense with variations of K_w in the specified regions of S_r. Assuming constant K_w for the specified regions of S_r is, indeed, assuming it as a step function of S_r that simply reflects the basic concept of the relation between K_w and S_r.

The right-hand side of Eq. (22) has previously derived in Eq. (14) and consequently we obtain

$$\frac{\rho_w K_w}{\gamma_w}\nabla^2 p_w=\rho_w\beta\hat{u}_{i,i}\frac{\partial}{\partial t}(p_a-p_w)+\rho_w[\alpha+\beta(\hat{p}_a-\hat{p}_w)]\frac{\partial}{\partial t}(u_{i,i}). \tag{23}$$

4. Laplace transform

One of the most common and straightforward methods for eliminating the time variable of a partial differential equation is to apply the Laplace transform. In this manner, after solving the differential equation in Laplace transform domain, one can obtain the time domain solution by applying an inverse Laplace transform on the Laplace transform domain solution. We remember that (Spiegel, 1965)

$$\begin{aligned}&\mathscr{L}[f(x,t),s]=\tilde{f}(x,s)=\int_0^\infty \mathrm{e}^{-st}f(x,t)\,\mathrm{d}t\\&\mathscr{L}\left(\frac{\partial}{\partial t}f,s,t\right)=s\mathscr{L}(f)-f_{(t=0)}\end{aligned} \tag{24}$$

and assuming

$$u_{i,(t=0)} = 0 \tag{25}$$

Eqs. (6), (20) and (23) will be reduced to Eqs. (26)–(28)

$$(\lambda + \mu)\tilde{u}_{j,ij} + \mu\tilde{u}_{i,jj} - \tilde{p}_{\mathrm{a},i} + D_{\mathrm{s}}(\tilde{p}_{\mathrm{a},i} - \tilde{p}_{\mathrm{w},i}) + \tilde{b}_i = 0, \tag{26}$$

$$\begin{aligned}\frac{\rho_\mathrm{a}K_\mathrm{a}}{\gamma_\mathrm{a}}\nabla^2\tilde{p}_\mathrm{a} + \frac{H\rho_\mathrm{a}K_\mathrm{w}}{\gamma_\mathrm{w}}\nabla^2\tilde{p}_\mathrm{w} = {}& \rho_\mathrm{a}s[1 - (\alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w}))(1 - H)]\tilde{u}_{i,i} \\ & - \rho_\mathrm{a}\hat{u}_{i,i}s\beta(1 - H)(\tilde{p}_\mathrm{a} - \tilde{p}_\mathrm{w}) - \rho_\mathrm{a}[1 - (\alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w}))(1 - H)]u_{i,i(t=0)} \\ & + \rho_\mathrm{a}\hat{u}_{i,i}\beta(1 - H)(p_{\mathrm{a}(t=0)} - p_{\mathrm{w}(t=0)}),\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\rho_\mathrm{w}K_\mathrm{w}}{\gamma_\mathrm{w}}\nabla^2\tilde{p}_\mathrm{w} = {}& \rho_\mathrm{w}s[\alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w})]\tilde{u}_{i,i} + \rho_\mathrm{w}s\hat{u}_{i,i}\beta(\tilde{p}_\mathrm{a} - \tilde{p}_\mathrm{w}) \\ & - \rho_\mathrm{w}[\alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w})]u_{i,i(t=0)} - \rho_\mathrm{w}\hat{u}_{i,i}\beta(p_{\mathrm{a}(t=0)} - p_{\mathrm{w}(t=0)}),\end{aligned} \tag{28}$$

where $\mathscr{L}$ is the Laplace transform operator, s is the Laplace transform parameter and the tilde denotes the variables in Laplace transform domain. Finally, one can simplify the above three equations in the forms of

$$c_{11}\tilde{u}_{j,ij} + c_{12}\tilde{u}_{i,jj} + c_{13}\tilde{p}_{\mathrm{a},i} + c_{14}\tilde{p}_{\mathrm{w},i} + c_{15} = 0, \tag{29}$$

$$c_{21}\tilde{u}_{i,i} + c_{22}\tilde{p}_\mathrm{a} + c_{23}\nabla^2\tilde{p}_\mathrm{a} + c_{24}\tilde{p}_\mathrm{w} + c_{25}\nabla^2\tilde{p}_\mathrm{w} + c_{26} = 0, \tag{30}$$

$$c_{31}\tilde{u}_{i,i} + c_{32}\tilde{p}_\mathrm{a} + c_{33}\nabla^2\tilde{p}_\mathrm{w} + c_{34}\tilde{p}_\mathrm{w} + c_{35} = 0 \quad i,j = \overline{1,2}, \tag{31}$$

where the c_{ij} coefficients are

$$\begin{aligned}
c_{11} &= \lambda + \mu \\
c_{12} &= \mu \\
c_{13} &= -1 + D_\mathrm{s} \\
c_{14} &= -D_\mathrm{s} \\
c_{15} &= \tilde{b}_i \\
c_{21} &= s\rho_\mathrm{a}[1 - (\alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w}))(1 - H)] \\
c_{22} &= -s\rho_\mathrm{a}\beta\hat{u}_{i,i}(1 - H) \\
c_{23} &= -\frac{\rho_\mathrm{a}K_\mathrm{a}}{\gamma_\mathrm{a}} \\
c_{24} &= s\rho_\mathrm{a}\beta\hat{u}_{i,i}(1 - H) \\
c_{25} &= -\frac{H\rho_\mathrm{a}K_\mathrm{w}}{\gamma_\mathrm{w}} \\
c_{26} &= -\rho_\mathrm{a}[1 - (\alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w}))(1 - H)]u_{i,i(t=0)} + \rho_\mathrm{a}\hat{u}_{i,i}\beta(1 - H)(p_{\mathrm{a}(t=0)} - p_{\mathrm{w}(t=0)}) \\
c_{31} &= s\rho_\mathrm{w}[\alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w})] \\
c_{32} &= s\rho_\mathrm{w}\beta\hat{u}_{i,i} \\
c_{33} &= -\frac{\rho_\mathrm{w}K_\mathrm{w}}{\gamma_\mathrm{w}} \\
c_{34} &= -s\rho_\mathrm{w}\beta\hat{u}_{i,i} \\
c_{35} &= -\rho_\mathrm{w}[\alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w})]u_{i,i(t=0)} - \rho_\mathrm{w}\beta(p_{\mathrm{a}(t=0)} - p_{\mathrm{w}(t=0)})\hat{u}_{i,i}.
\end{aligned} \tag{32}$$

5. Green's functions

One may write the differential equations (29)–(31) in the following matrix form:

$$[C_{ij}] \times \vec{\omega} = \vec{f}, \tag{33}$$

where $C_{ij} = c_{ij} \times d_{ij}$ and

$$\begin{aligned} &\omega_i = \tilde{u}_i \quad i = \overline{1,2} \\ &\omega_3 = \tilde{p}_{\mathrm{a}} \\ &\omega_4 = \tilde{p}_{\mathrm{w}} \end{aligned} \tag{34}$$

and

$$\begin{aligned} &f_i = -\tilde{b}_i \quad i = \overline{1,2} \\ &f_3 = -c_{26} \\ &f_4 = -c_{35} \end{aligned} \tag{35}$$

in which d_{ij} are the differential operators.

The physical interpretation of Green's function, fundamental solution or kernel of a differential equation is a potential function $p(x,\xi)$ in the point x of the domain that has been resulted from an excitation $e(\xi)$ in excitation point ξ. This excitation may be the Dirac delta function or the unit impact load. On the other hand, the fundamental solution is the simplest solution of the differential equation that is due to a unit and instantly impact in a domain with infinite boundaries.

Therefore, one may assume a unit point load $\delta(x)$ instead of the right-hand side of the differential equation. The most common and straightforward method for deriving the Green's functions of a system of differential equations, is the Kupradze (Kupradze et al., 1979) or Hörmander's method (Hörmander, 1963). According to this method, the problem is to find the function $G = [\tilde{g}_{ij}]$ which satisfies the equation

$$[C_{ik}][\tilde{g}_{kj}] + [I]\delta(x) = 0, \tag{36}$$

where $[I]$ is the identity matrix. Also from the matrix algebra we know that

$$[C_{ik}][C^*_{kj}] = [I]\det(C_{ij}). \tag{37}$$

If there is a scalar function that satisfies Eq. (38)

$$\det(C_{ij})\varphi + \delta(x) = 0, \tag{38}$$

substituting from Eq. (37) into Eq. (38) and multiplying by $[I]$, one may obtain

$$[C_{ik}][C^*_{kj}]\varphi + [I]\delta(x) = 0. \tag{39}$$

The comparison of Eqs. (36) and (39) leads to

$$[\tilde{g}_{kj}] = [C^*_{kj}]\varphi. \tag{40}$$

Indeed, the problem has been reduced to finding a scalar function φ that satisfies Eq. (38) and the computation of cofactor matrix $[C^*_{kj}]$.

Computing the determinant of the $[C_{ij}]$ matrix, Eq. (38) under Laplace transform is as follows:

$$(D_1\nabla^8 + D_2\nabla^6 + D_3\nabla^4)\varphi + \frac{1}{s}\delta(x) = 0, \tag{41}$$

where $\nabla^{2n} = (\nabla^2)^n$ represents n occurrence(s) of the Laplacian operator. Also D_1, D_2 and D_3 are

$$\begin{aligned} D_1 &= c_{12}(c_{11}+c_{12})c_{23}c_{33} \\ D_2 &= c_{12}(-c_{14}c_{23}c_{31}+c_{13}(c_{25}c_{31}-c_{21}c_{33})-(c_{11}+c_{12})(c_{25}c_{32}-c_{22}c_{33}-c_{23}c_{34})) \\ D_3 &= c_{12}(c_{13}(c_{24}c_{31}-c_{21}c_{34})+c_{14}(c_{21}c_{32}-c_{22}c_{31})-(c_{11}+c_{12})(c_{24}c_{32}-c_{22}c_{34})). \end{aligned} \tag{42}$$

Now, one can write Eq. (41) in the following form:

$$\left(\nabla^4 + \frac{D_2}{D_1}\nabla^2 + \frac{D_3}{D_1}\right)[D_1 s\nabla^4(\varphi)] + \delta(x) = 0 \tag{43}$$

defining an interim function Φ as

$$\Phi = D_1 s\nabla^4(\varphi) \tag{44}$$

Eq. (43) will be changed to the form of

$$(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2)\Phi + \delta(x) = 0, \tag{45}$$

where λ_1^2 and λ_2^2 are

$$\lambda_{1,2}^2 = \frac{-D_2 \pm \sqrt{D_2^2 - 4D_1D_3}}{2D_1}. \tag{46}$$

Eq. (45) may be written as either of the two Eqs. (47) and (48)

$$\begin{aligned} &(\nabla^2 - \lambda_1^2)\Phi_1 + \delta(x) = 0 \\ &\Phi_1 = (\nabla^2 - \lambda_2^2)\Phi \end{aligned} \tag{47}$$

$$\begin{aligned} &(\nabla^2 - \lambda_2^2)\Phi_2 + \delta(x) = 0 \\ &\Phi_2 = (\nabla^2 - \lambda_1^2)\Phi \end{aligned} \tag{48}$$

The above differential equations are of the familiar Helmholtz type. The Green's function of Helmholtz differential equations for an only r-dependent fully symmetric *two-dimensional* domain is (Arfken and Weber, 2001 and Ocendon et al., 1999)

$$\Phi_i = \frac{K_0(\lambda_i r)}{2\pi} \quad i = \overline{1,2}, \tag{49}$$

where K_n is the modified Bessel function of order n. However, by subtracting Eq. (47) from (48), one can obtain

$$\Phi_2 - \Phi_1 = (\lambda_2^2 - \lambda_1^2)\Phi \tag{50}$$

and therefore

$$\Phi = \frac{\Phi_2 - \Phi_1}{(\lambda_2^2 - \lambda_1^2)} = \frac{K_0(\lambda_2 r) - K_0(\lambda_1 r)}{2\pi(\lambda_2^2 - \lambda_1^2)}. \tag{51}$$

Applying two times the *two-dimensional* inverse Laplacian operator (Spiegel, 1999)

$$\nabla^{-2}(\vartheta) = \int_r \left(r^{-1}\int_r (r\vartheta)\mathrm{d}r\right)\mathrm{d}r \tag{52}$$

one may obtain the φ function as

$$\varphi(r,s)=\frac{1}{2\pi D_1 s(\lambda_2^2-\lambda_1^2)}\left(\frac{K_0(\lambda_2 r)}{\lambda_2^4}-\frac{K_0(\lambda_1 r)}{\lambda_1^4}\right). \tag{53}$$

The $[\tilde{g}_{ij}]$ Green's functions or cofactor matrix components $[C_{ij}^*]$ are

$$\begin{aligned}
\tilde{g}_{ij}&=[\delta_{ij}(F_{11}\nabla^6+F_{12}\nabla^4+F_{13}\nabla^2)+(F_{21}\nabla^4\partial_i\partial_j+F_{22}\nabla^2\partial_i\partial_j+F_{23}\partial_i\partial_j)]\varphi\\
\tilde{g}_{i3}&=(F_{31}\nabla^4\partial_i+F_{32}\nabla^2\partial_i)\varphi\\
\tilde{g}_{i4}&=(F_{41}\nabla^4\partial_i+F_{42}\nabla^2\partial_i)\varphi\\
\tilde{g}_{3i}&=(F_{51}\nabla^4\partial_i+F_{52}\nabla^2\partial_i)\varphi\\
\tilde{g}_{4i}&=(F_{61}\nabla^4\partial_i+F_{62}\nabla^2\partial_i)\varphi\\
\tilde{g}_{33}&=(F_{71}\nabla^6+F_{72}\nabla^4)\varphi\\
\tilde{g}_{34}&=(F_{73}\nabla^6+F_{74}\nabla^4)\varphi\\
\tilde{g}_{43}&=(F_{75}\nabla^4)\varphi\\
\tilde{g}_{44}&=(F_{76}\nabla^6+F_{77}\nabla^4)\varphi \quad i,j=\overline{1,2}.
\end{aligned} \tag{54}$$

The F_{ij} coefficients are presented in Appendix A.

5.1. Green's functions in Laplace transform domain

Now by substituting the φ function from Eq. (53) into Eq. (54) and defining intermediate Γ_i functions

$$\begin{aligned}
\Gamma_1&=K_{11}\Omega_{11}+K_{12}\Omega_{12}+K_{13}\Omega_{13}\\
\Gamma_2&=K_{21}\Omega_{31}+K_{22}\Omega_{32}+K_{23}\Omega_{33}\\
\Gamma_3&=K_{21}\Omega_{11}+K_{22}\Omega_{12}+K_{23}\Omega_{13}
\end{aligned} \tag{55}$$

one can obtain the Green's functions in Laplace transform domain as follow:

$$\begin{aligned}
\tilde{g}_{ij}&=\delta_{ij}\Gamma_1+\frac{1}{r^3}(2x_ix_j-\delta_{ij}r^2)\Gamma_2+\frac{x_ix_j}{r^2}\Gamma_3\\
\tilde{g}_{i3}&=-\frac{x_i}{r}(K_{31}\Omega_{31}+K_{32}\Omega_{32})\\
\tilde{g}_{i4}&=-\frac{x_i}{r}(K_{41}\Omega_{31}+K_{42}\Omega_{32})\\
\tilde{g}_{3i}&=-\frac{x_i}{r}(K_{51}\Omega_{21}+K_{52}\Omega_{22})\\
\tilde{g}_{4i}&=-\frac{x_i}{r}(K_{61}\Omega_{21}+K_{62}\Omega_{22})\\
\tilde{g}_{33}&=K_{71}\Omega_{11}+K_{72}\Omega_{12}\\
\tilde{g}_{34}&=K_{73}\Omega_{11}+K_{74}\Omega_{12}\\
\tilde{g}_{43}&=K_{75}\Omega_{12}\\
\tilde{g}_{44}&=K_{76}\Omega_{11}+K_{77}\Omega_{12} \quad i,j=\overline{1,2}.
\end{aligned} \tag{56}$$

The above Green's functions are presented in extended form in Appendix D. It is evident from the relationships in Appendix D that $\tilde{g}_{3i} = s\tilde{g}_{i3}$ and $\tilde{g}_{4i} = s\tilde{g}_{i4}$ as have been emphasized by Chen (1994a). Furthermore, K_{ij} coefficients and Ω_{ij} intermediate functions are shown in Appendices B and C, respectively.

5.2. Green's functions in the time domain

In order to apply the inverse Laplace transform to the Laplace transform domain Green's functions, we need to find the inverse Laplace transforms of the following terms:

$$\frac{K_0(r\lambda_2)}{(\lambda_2^2-\lambda_1^2)},\quad \frac{\lambda_2^2K_0(r\lambda_2)}{s(\lambda_2^2-\lambda_1^2)},\quad \frac{sK_0(r\lambda_2)}{\lambda_2^2(\lambda_2^2-\lambda_1^2)},\quad \frac{K_1(r\lambda_2)\lambda_2}{(\lambda_2^2-\lambda_1^2)},\quad \frac{sK_1(r\lambda_2)}{\lambda_2(\lambda_2^2-\lambda_1^2)},$$
$$\frac{K_1(r\lambda_2)\lambda_2}{s(\lambda_2^2-\lambda_1^2)},\quad \frac{K_1(r\lambda_2)}{\lambda_2(\lambda_2^2-\lambda_1^2)},\quad \frac{sK_1(r\lambda_2)}{\lambda_2^3(\lambda_2^2-\lambda_1^2)}, \tag{57}$$

where

$$\begin{aligned}\lambda_1 &= \sqrt{m_1}\sqrt{s}\\ \lambda_2 &= \sqrt{m_2}\sqrt{s}\\ \lambda_2^2-\lambda_1^2 &= m_3 s\end{aligned} \tag{58}$$

and the m_i coefficients in Eq. (58) are

$$m_{1,2} = \frac{-\dfrac{D_2}{s} \pm \sqrt{\dfrac{D_2^2-4D_1D_3}{s^2}}}{2D_1} \qquad m_3 = m_2 - m_1 \tag{59}$$

According to the Laplace transform references, the inverse Laplace transform of the following terms are available (Abramowitz and Stegun, 1965 and Spiegel, 1965):

$$\frac{K_0(a\sqrt{s})}{s},\quad \frac{K_1(a\sqrt{s})}{\sqrt{s}},\quad \frac{K_1(a\sqrt{s})}{s\sqrt{s}}. \tag{60}$$

The inverse Laplace transforms of the terms in Eq. (60) are shown as $\Lambda_{ij}[a,t]$ in Appendix E. Now, using the inverse Laplace transforms $\Lambda_{ij}[a,t]$, we can obtain the inverse Laplace transforms of the Green's functions in Eq. (56). For this purpose, the intermediate functions $\Psi_{ij}[r,t]$ are defined in Appendix F. Using the K_{ij} coefficients and the intermediate functions $\Psi_{ij}[r,t]$, we are able to derive the Green's functions in the time domain, which are shown in Eq. (62). Defining the Θ_i intermediate functions as follow:

$$\begin{aligned}\Theta_1 &= K_{11}\Psi_{11}[r,t] + K_{12}\Psi_{12}[r,t] + K_{13}\Psi_{13}[r,t]\\ \Theta_2 &= K_{21}\Psi_{31}[r,t] + K_{22}\Psi_{32}[r,t] + K_{23}\Psi_{33}[r,t]\\ \Theta_3 &= K_{21}\Psi_{11}[r,t] + K_{22}\Psi_{12}[r,t] + K_{23}\Psi_{13}[r,t]\end{aligned} \tag{61}$$

the Green's functions are

$$\begin{aligned}g_{ij}[r,x_i,x_j,t] &= \delta_{ij}\Theta_1 + \frac{1}{r^3}(2x_ix_j - \delta_{ij}r^2)\Theta_2 + \frac{x_ix_j}{r^2}\Theta_3\\ g_{i3}[r,x_i,t] &= -\frac{x_i}{r}(K_{31}\Psi_{31}[r,t] + K_{32}\Psi_{32}[r,t])\\ g_{i4}[r,x_i,t] &= -\frac{x_i}{r}(K_{41}\Psi_{31}[r,t] + K_{42}\Psi_{32}[r,t])\end{aligned}$$

$$
\begin{aligned}
g_{3i}[r,x_i,t] &= -\frac{x_i}{r}(K_{51}\Psi_{21}[r,t]+K_{52}\Psi_{22}[r,t])\\
g_{4i}[r,x_i,t] &= -\frac{x_i}{r}(K_{61}\Psi_{21}[r,t]+K_{62}\Psi_{22}[r,t])\\
g_{33}[r,t] &= K_{71}\Psi_{11}[r,t]+K_{72}\Psi_{12}[r,t]\\
g_{34}[r,t] &= K_{73}\Psi_{11}[r,t]+K_{74}\Psi_{12}[r,t]\\
g_{43}[r,t] &= K_{75}\Psi_{12}[r,t]\\
g_{44}[r,t] &= K_{76}\Psi_{11}[r,t]+K_{77}\Psi_{12}[r,t] \quad i,j=\overline{1,2}.
\end{aligned} \tag{62}
$$

6. Verification

Since the solutions are presented here for the first time there is no chance for them to be compared with other corresponding results to find the probable differences and their reasons. However, for the solutions (mathematical model) to be verified mathematically (Babuska and Oden, 2004), we can show that, for example if the conditions approach to the poroelastostatic case, the corresponding Green's functions will approach to the poroelastostatic Green's functions {neglecting dissolved air in water and the suction effect (i.e. $H = D_s = 0$)}. For this purpose and considering Eqs. (29)–(31), we only need to substitute the coefficients of terms that have time variations, such as $\widehat{S}_r$ and $\hat{n}$, with zero. This means to substitute the terms ξ (or $\widehat{S}_r$) and η (or $(1-\widehat{S}_r)$) and also $\hat{u}_{i,i}$ in K_{ij} statements, with zero. Therefore, only the following coefficients will remain non-vanishing:

$$
\begin{aligned}
K_{11} &= \frac{1}{2\pi\mu}\\
K_{21} &= -\frac{\lambda+\mu}{2\pi\mu(\lambda+2\mu)}\\
K_{31} &= -\frac{\gamma_a}{2\pi(\lambda+2\mu)K_a\rho_a}\\
K_{71} &= -\frac{\gamma_a}{2\pi K_a\rho_a}\\
K_{76} &= -\frac{\gamma_w}{2\pi K_w\rho_w}.
\end{aligned} \tag{63}
$$

Among the Ω_{ij} terms in Laplace transform Green's functions in Appendix C, those have non-zero terms are

$$
\begin{aligned}
\Omega_{11} &= \frac{1}{s(\lambda_2^2-\lambda_1^2)}(K_0(\lambda_2 r)\lambda_2^2 - K_0(\lambda_1 r)\lambda_1^2)\\
\Omega_{31} &= \frac{1}{s(\lambda_2^2-\lambda_1^2)}(K_1(\lambda_2 r)\lambda_2 - K_1(\lambda_1 r)\lambda_1).
\end{aligned} \tag{64}
$$

By substituting the terms ξ (or $\widehat{S}_r$) and also $\hat{u}_{i,i}$ with zero, all the m_i terms and subsequently λ_1 and λ_2 will vanish. Therefore we necessarily shall find the limits of Ω_{11} and Ω_{31} while λ_1 and λ_2 approach zero. These limits are

$$
\begin{aligned}
&\lim_{\lambda_1,\lambda_2\to 0}\{\Omega_{11}\} = -\frac{\ln(r)}{s}\\
&\lim_{\lambda_1,\lambda_2\to 0}\{\Omega_{31}\} = \frac{r}{2s}\left(\frac{1}{2}+\ln(r)\right).
\end{aligned}
\tag{65}
$$

Furthermore, all of the Ω_{ij} terms in the Green's functions in Laplace transform domain that have zero coefficients, have no limits and consequently, their coefficients being zero seems to be normal or inevitable.

Using the limits in Eq. (65), the Green's functions in Laplace transform domain, after some simplifications will be obtained as follow:

$$
\begin{aligned}
&\tilde{g}_{ij} = \frac{[(\lambda+\mu)-2(\lambda+3\mu)\ln(r)]r^2\delta_{ij}+2(\lambda+\mu)x_ix_j}{8\pi r^2 s\mu(\lambda+2\mu)}\\
&\tilde{g}_{3i} = \tilde{g}_{4i} = 0\\
&\tilde{g}_{i3} = \frac{\gamma_a x_i[1+2\ln(r)]}{4\pi s(\lambda+2\mu)K_a\rho_a}\\
&\tilde{g}_{i4} = 0\\
&\tilde{g}_{33} = \frac{\gamma_a \ln(r)}{2\pi s K_a\rho_a}\\
&\tilde{g}_{34} = \tilde{g}_{43} = 0\\
&\tilde{g}_{44} = \frac{\gamma_w \ln(r)}{2\pi s K_w\rho_w} \quad i,j=\overline{1,2}
\end{aligned}
\tag{66}
$$

and their corresponding terms in time domain are

$$
\begin{aligned}
&g_{ij} = \frac{[(\lambda+\mu)-2(\lambda+3\mu)\ln(r)]r^2\delta_{ij}+2(\lambda+\mu)x_ix_j}{8\pi r^2 \mu(\lambda+2\mu)}\\
&g_{3i} = g_{4i} = 0\\
&g_{i3} = \frac{\gamma_a x_i[1+2\ln(r)]}{4\pi(\lambda+2\mu)K_a\rho_a}\\
&g_{i4} = 0\\
&g_{33} = \frac{\gamma_a \ln(r)}{2\pi K_a\rho_a}\\
&g_{34} = g_{43} = 0\\
&g_{44} = \frac{\gamma_w \ln(r)}{2\pi K_w\rho_w} \quad i,j=\overline{1,2}
\end{aligned}
\tag{67}
$$

that are exactly the poroelastostatic Green's functions (Banerjee, 1994; Gatmiri and Jabbari, 2004a).

Also, from mathematical point of view, one can determine the singularities of the Green's functions. Since for the common physical parameters for unsaturated soils

$$
\begin{aligned}
&\lim_{r\to 0}\{\Psi_{ij}\} = \infty\\
&\lim_{r\to \infty}\{\Psi_{ij}\} = 0 \quad i,j=\overline{1,3}
\end{aligned}
\tag{68}
$$

the singularity of all the Green's functions is only at $r = 0$.

7. Conclusion

In this research the closed form *two-dimensional* Green's functions of the governing equations of unsaturated soils, including equilibrium equations with linear elastic constitutive equations and two equations of air and water transfer have been derived in Laplace transform and time domains for the first time. For verification of the resulted Green's functions, it has been demonstrated that if the conditions approach to poroelastostatic case, the Green's functions will approach to poroelastostatic Green's functions exactly.

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Appendix A

F_{ij} coefficients:

$$F_{11} = (c_{11} + c_{12})c_{23}c_{33}$$

$$F_{12} = -c_{14}c_{23}c_{31} + c_{13}(c_{25}c_{31} - c_{21}c_{33}) - (c_{11} + c_{12})(c_{25}c_{32} - c_{22}c_{33} - c_{23}c_{34})$$

$$F_{13} = c_{14}(c_{21}c_{32} - c_{22}c_{31}) + c_{13}(c_{24}c_{31} - c_{21}c_{34}) - (c_{11} + c_{12})(c_{24}c_{32} - c_{22}c_{34})$$

$$F_{21} = -c_{11}c_{23}c_{33}$$

$$F_{22} = c_{14}c_{23}c_{31} + c_{13}(c_{21}c_{33} - c_{25}c_{31}) + c_{11}(c_{25}c_{32} - c_{22}c_{33} - c_{23}c_{34})$$

$$F_{23} = c_{14}(c_{22}c_{31} - c_{21}c_{32}) + c_{13}(c_{21}c_{34} - c_{24}c_{31}) + c_{11}(c_{24}c_{32} - c_{22}c_{34})$$

$$F_{31} = -c_{12}c_{13}c_{33} \qquad F_{32} = c_{12}(c_{14}c_{32} - c_{13}c_{34})$$

$$F_{41} = c_{12}(c_{13}c_{25} - c_{14}c_{23}) \qquad F_{42} = c_{12}(c_{13}c_{24} - c_{14}c_{22})$$

$$F_{51} = c_{12}(c_{25}c_{31} - c_{21}c_{33}) \qquad F_{52} = c_{12}(c_{24}c_{31} - c_{21}c_{34})$$

$$F_{61} = -c_{12}c_{23}c_{31} \qquad F_{62} = c_{12}(c_{21}c_{32} - c_{22}c_{31})$$

$$F_{71} = c_{12}(c_{11} + c_{12})c_{33} \qquad F_{72} = c_{12}(-c_{14}c_{31} + (c_{11} + c_{12})c_{34})$$

$$F_{73} = -c_{12}(c_{11} + c_{12})c_{25} \qquad F_{74} = -c_{12}(-c_{14}c_{21} + (c_{11} + c_{12})c_{24})$$

$$F_{75} = -c_{12}(-c_{13}c_{31} + (c_{11} + c_{12})c_{32}) \qquad F_{76} = c_{12}(c_{11} + c_{12})c_{23}$$

$$F_{77} = c_{12}(-c_{13}c_{21} + (c_{11} + c_{12})c_{22})$$

Appendix B

K_{ij} coefficients:

$$\xi = \alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w}) \quad \eta = 1 - \xi(1 - H)$$

$$K_{11} = \frac{F_{11}}{2\pi D_1} = \frac{1}{2\pi\mu}$$

$$K_{12} = \frac{F_{12}}{2\pi D_1 s} = \frac{\beta(\lambda + 2\mu)(K_\mathrm{a}\gamma_\mathrm{w} + K_\mathrm{w}\gamma_\mathrm{a})\hat{u}_{i,i} + K_\mathrm{w}\gamma_\mathrm{a}(1 - \xi)(-1 + D_\mathrm{s}) - K_\mathrm{a}\gamma_\mathrm{w}\xi D_\mathrm{s}}{2\pi\mu(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}}$$

$$K_{13} = \frac{F_{13}}{2\pi D_1 s^2} = -\frac{\beta\gamma_\mathrm{a}\gamma_\mathrm{w}\hat{u}_{i,i}}{2\pi\mu(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}}$$

$$K_{21} = \frac{F_{21}}{2\pi D_1} = -\frac{\lambda + \mu}{2\pi\mu(\lambda + 2\mu)}$$

$$K_{22} = \frac{F_{22}}{2\pi D_1 s} = -\frac{\beta(\lambda + \mu)(K_\mathrm{a}\gamma_\mathrm{w} + K_\mathrm{w}\gamma_\mathrm{a})\hat{u}_{i,i} + K_\mathrm{w}\gamma_\mathrm{a}(1 - \xi)(-1 + D_\mathrm{s}) - K_\mathrm{a}\gamma_\mathrm{w}\xi D_\mathrm{s}}{2\pi\mu(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}}$$

$$K_{23} = \frac{F_{23}}{2\pi D_1 s^2} = -K_{13}$$

$$K_{31} = \frac{F_{31}}{2\pi D_1} = \frac{\gamma_\mathrm{a}(-1 + D_\mathrm{s})}{2\pi(\lambda + 2\mu)K_\mathrm{a}\rho_\mathrm{a}} \qquad K_{32} = \frac{F_{32}}{2\pi D_1 s} = -\frac{\beta\gamma_\mathrm{a}\gamma_\mathrm{w}\hat{u}_{i,i}}{2\pi(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{a}}$$

$$K_{41} = \frac{F_{41}}{2\pi D_1} = -\frac{H(-1 + D_\mathrm{s})K_\mathrm{w}\gamma_\mathrm{a} + D_\mathrm{s}K_\mathrm{a}\gamma_\mathrm{w}}{2\pi(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{w}} \qquad K_{42} = \frac{F_{42}}{2\pi D_1 s} = \frac{(-1 + H)\beta\gamma_\mathrm{a}\gamma_\mathrm{w}\hat{u}_{i,i}}{2\pi(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{w}}$$

$$K_{51} = \frac{F_{51}}{2\pi D_1 s} = \frac{(1 - \xi)\gamma_\mathrm{a}}{2\pi(\lambda + 2\mu)K_\mathrm{a}} \qquad K_{52} = \frac{F_{52}}{2\pi D_1 s^2} = \mu K_{23}$$

$$K_{61} = \frac{F_{61}}{2\pi D_1 s} = \frac{\gamma_\mathrm{w}\xi}{2\pi(\lambda + 2\mu)K_\mathrm{w}} \qquad K_{62} = \frac{F_{62}}{4\pi D_1 s^2} = K_{52}$$

$$K_{71} = \frac{F_{71}}{2\pi D_1} = -\frac{\gamma_\mathrm{a}}{2\pi K_\mathrm{a}\rho_\mathrm{a}} \qquad K_{72} = \frac{F_{72}}{2\pi D_1 s} = \frac{\gamma_\mathrm{a}\gamma_\mathrm{w}(\xi D_\mathrm{s} - \beta(\lambda + 2\mu)\hat{u}_{i,i})}{2\pi(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{a}}$$

$$K_{73} = \frac{F_{73}}{2\pi D_1} = \frac{H\gamma_\mathrm{a}}{2\pi K_\mathrm{a}\rho_\mathrm{w}}$$

$$K_{74} = \frac{F_{74}}{2\pi D_1 s} = -\frac{\gamma_\mathrm{a}\gamma_\mathrm{w}(\eta D_\mathrm{s} + (1 - H)\beta\hat{u}_{i,i}(\lambda + 2\mu))}{2\pi(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{w}}$$

$$K_{75} = \frac{F_{75}}{2\pi D_1 s} = -\frac{\gamma_\mathrm{a}\gamma_\mathrm{w}(\xi(1 - D_\mathrm{s}) + \beta\hat{u}_{i,i}(\lambda + 2\mu))}{2\pi(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{a}}$$

$$K_{76} = \frac{F_{76}}{2\pi D_1} = -\frac{\gamma_\mathrm{w}}{2\pi K_\mathrm{w}\rho_\mathrm{w}}$$

$$K_{77} = \frac{F_{77}}{2\pi D_1 s} = \frac{\gamma_\mathrm{a}\gamma_\mathrm{w}(\eta(1 - D_\mathrm{s}) - (1 - H)\beta\hat{u}_{i,i}(\lambda + 2\mu))}{2\pi(\lambda + 2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{w}}$$

Appendix C

The intermediate functions Ω_{ij}:

$$\Omega_{11} = \frac{K_0(r\lambda_2)\lambda_2^2 - K_0(r\lambda_1)\lambda_1^2}{s(\lambda_2^2 - \lambda_1^2)} \qquad \Omega_{12} = \frac{K_0(r\lambda_2) - K_0(r\lambda_1)}{(\lambda_2^2 - \lambda_1^2)}$$

$$\Omega_{13} = \frac{s}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{K_0(r\lambda_2)}{\lambda_2^2} - \frac{K_0(r\lambda_1)}{\lambda_1^2}\right)$$

$$\Omega_{21} = \frac{1}{(\lambda_2^2 - \lambda_1^2)}(K_1(r\lambda_2)\lambda_2 - K_1(r\lambda_1)\lambda_1) \qquad \Omega_{22} = \frac{s}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{K_1(r\lambda_2)}{\lambda_2} - \frac{K_1(r\lambda_1)}{\lambda_1}\right)$$

$$\Omega_{31} = \frac{1}{s(\lambda_2^2 - \lambda_1^2)}(K_1(r\lambda_2)\lambda_2 - K_1(r\lambda_1)\lambda_1) \qquad \Omega_{32} = \frac{1}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{K_1(r\lambda_2)}{\lambda_2} - \frac{K_1(r\lambda_1)}{\lambda_1}\right)$$

$$\Omega_{33} = \frac{s}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{K_1(r\lambda_2)}{\lambda_2^3} - \frac{K_1(r\lambda_1)}{\lambda_1^3}\right)$$

Appendix D

The Green's functions in Laplace transform domain:

$$\begin{aligned}
\tilde{g}_{ij} = {} & \delta_{ij}K_{11}\frac{K_0(r\lambda_2)\lambda_2^2 - K_0(r\lambda_1)\lambda_1^2}{s(\lambda_2^2 - \lambda_1^2)} + \delta_{ij}K_{12}\frac{K_0(r\lambda_2) - K_0(r\lambda_1)}{(\lambda_2^2 - \lambda_1^2)} + \delta_{ij}K_{13}\frac{s}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{K_0(r\lambda_2)}{\lambda_2^2} - \frac{K_0(r\lambda_1)}{\lambda_1^2}\right) \\
& + K_{21}\frac{1}{r^3 s(\lambda_2^2 - \lambda_1^2)}\left[(2x_ix_j - \delta_{ij}r^2)(K_1(r\lambda_2)\lambda_2 - K_1(r\lambda_1)\lambda_1) + x_ix_jr(K_0(r\lambda_2)\lambda_2^2 - K_0(r\lambda_1)\lambda_1^2)\right] \\
& + K_{22}\frac{1}{r^3(\lambda_2^2 - \lambda_1^2)}\left[(2x_ix_j - \delta_{ij}r^2)\left(\frac{K_1(r\lambda_2)}{\lambda_2} - \frac{K_1(r\lambda_1)}{\lambda_1}\right) + x_ix_jr(K_0(r\lambda_2) - K_0(r\lambda_1))\right] \\
& + K_{23}\frac{s}{r^3(\lambda_2^2 - \lambda_1^2)}\left[(2x_ix_j - \delta_{ij}r^2)\left(\frac{K_1(r\lambda_2)}{\lambda_2^3} - \frac{K_1(r\lambda_1)}{\lambda_1^3}\right) + x_ix_jr\left(\frac{K_0(r\lambda_2)}{\lambda_2^2} - \frac{K_0(r\lambda_1)}{\lambda_1^2}\right)\right]
\end{aligned}$$

$$\tilde{g}_{i3} = -\frac{K_{31}x_i}{r}\frac{1}{s(\lambda_2^2 - \lambda_1^2)}(K_1(r\lambda_2)\lambda_2 - K_1(r\lambda_1)\lambda_1) - \frac{K_{32}x_i}{r}\frac{1}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{K_1(r\lambda_2)}{\lambda_2} - \frac{K_1(r\lambda_1)}{\lambda_1}\right)$$

$$\tilde{g}_{i4} = -\frac{K_{41}x_i}{r}\frac{1}{s(\lambda_2^2 - \lambda_1^2)}(K_1(r\lambda_2)\lambda_2 - K_1(r\lambda_1)\lambda_1) - \frac{K_{42}x_i}{r}\frac{1}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{K_1(r\lambda_2)}{\lambda_2} - \frac{K_1(r\lambda_1)}{\lambda_1}\right)$$

$$\tilde{g}_{3i} = -\frac{K_{51}x_i}{r}\frac{1}{(\lambda_2^2 - \lambda_1^2)}(K_1(r\lambda_2)\lambda_2 - K_1(r\lambda_1)\lambda_1) - \frac{K_{52}x_i}{r}\frac{s}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{K_1(r\lambda_2)}{\lambda_2} - \frac{K_1(r\lambda_1)}{\lambda_1}\right)$$

$$\tilde{g}_{4i} = -\frac{K_{61}x_i}{r}\frac{1}{(\lambda_2^2 - \lambda_1^2)}(K_1(r\lambda_2)\lambda_2 - K_1(r\lambda_1)\lambda_1) - \frac{K_{62}x_i}{r}\frac{s}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{K_1(r\lambda_2)}{\lambda_2} - \frac{K_1(r\lambda_1)}{\lambda_1}\right)$$

$$\tilde{g}_{33} = K_{71}\frac{K_0(r\lambda_2)\lambda_2^2 - K_0(r\lambda_1)\lambda_1^2}{s(\lambda_2^2 - \lambda_1^2)} + K_{72}\frac{K_0(r\lambda_2) - K_0(r\lambda_1)}{(\lambda_2^2 - \lambda_1^2)}$$

$$\tilde{g}_{34} = K_{73}\frac{K_0(r\lambda_2)\lambda_2^2 - K_0(r\lambda_1)\lambda_1^2}{s(\lambda_2^2 - \lambda_1^2)} + K_{74}\frac{K_0(r\lambda_2) - K_0(r\lambda_1)}{(\lambda_2^2 - \lambda_1^2)}$$

$$\tilde{g}_{43} = K_{75}\frac{K_0(r\lambda_2) - K_0(r\lambda_1)}{(\lambda_2^2 - \lambda_1^2)}$$

$$\tilde{g}_{44} = K_{76}\frac{K_0(r\lambda_2)\lambda_2^2 - K_0(r\lambda_1)\lambda_1^2}{s(\lambda_2^2 - \lambda_1^2)} + K_{77}\frac{K_0(r\lambda_2) - K_0(r\lambda_1)}{(\lambda_2^2 - \lambda_1^2)}$$

Appendix E

The inverse Laplace transforms and intermediate functions $\Lambda_{ij}[a,t]$:

$$\Lambda_0[a,t] = \mathscr{L}^{-1}\left\{\frac{K_0(a\sqrt{s})}{s}\right\} = \frac{1}{2}\Gamma\left(0,\frac{a^2}{4t}\right)$$

$$\Lambda_{11}[a,t] = \mathscr{L}^{-1}\left\{\frac{K_1(a\sqrt{s})}{\sqrt{s}}\right\} = \frac{1}{a}\mathrm{e}^{-\frac{a^2}{4t}}$$

$$\Lambda_{12}[a,t] = \mathscr{L}^{-1}\left\{\frac{K_1(a\sqrt{s})}{s\sqrt{s}}\right\} = \mathrm{e}^{-\frac{a^2}{4t}}t - \frac{1}{4}a^2\Gamma\left(0,\frac{a^2}{4t}\right)$$

in which

$$\Gamma(a,x) = \int_x^\infty t^{a-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

and $K_i(x)$ is the *modified Bessel function* of order i.

Appendix F

The intermediate functions $\Psi_{ij}[r,t]$:

$$\Psi_{11}[r,t] = \mathscr{L}^{-1}\{\Omega_{11}\} = \frac{1}{m_3}(m_2\Lambda_0[r\sqrt{m_2},t] - m_1\Lambda_0[r\sqrt{m_1},t])$$

$$\Psi_{12}[r,t] = \mathscr{L}^{-1}\{\Omega_{12}\} = \frac{1}{m_3}(\Lambda_0[r\sqrt{m_2},t] - \Lambda_0[r\sqrt{m_1},t])$$

$$\Psi_{13}[r,t] = \mathscr{L}^{-1}\{\Omega_{13}\} = \frac{1}{m_3}\left(\frac{1}{m_2}\Lambda_0[r\sqrt{m_2},t] - \frac{1}{m_1}\Lambda_0[r\sqrt{m_1},t]\right)$$

$$\Psi_{21}[r,t] = \mathscr{L}^{-1}\{\Omega_{21}\} = \frac{1}{m_3}(\sqrt{m_2}\Lambda_{11}[r\sqrt{m_2},t] - \sqrt{m_1}\Lambda_{11}[r\sqrt{m_1},t])$$

$$\Psi_{22}[r,t] = \mathscr{L}^{-1}\{\Omega_{22}\} = \frac{1}{m_3}\left(\frac{1}{\sqrt{m_2}}\Lambda_{11}[r\sqrt{m_2},t] - \frac{1}{\sqrt{m_1}}\Lambda_{11}[r\sqrt{m_1},t]\right)$$

$$\Psi_{31}[r,t] = \mathscr{L}^{-1}\{\Omega_{31}\} = \frac{1}{m_3}(\sqrt{m_2}\Lambda_{12}[r\sqrt{m_2},t] - \sqrt{m_1}\Lambda_{12}[r\sqrt{m_1},t])$$

$$\Psi_{32}[r,t] = \mathscr{L}^{-1}\{\Omega_{32}\} = \frac{1}{m_3}\left(\frac{1}{\sqrt{m_2}}\Lambda_{12}[r\sqrt{m_2},t] - \frac{1}{\sqrt{m_1}}\Lambda_{12}[r\sqrt{m_1},t]\right)$$

$$\Psi_{33}[r,t] = \mathscr{L}^{-1}\{\Omega_{33}\} = \frac{1}{m_3}\left(\frac{1}{m_2\sqrt{m_2}}\Lambda_{12}[r\sqrt{m_2},t] - \frac{1}{m_1\sqrt{m_1}}\Lambda_{12}[r\sqrt{m_1},t]\right)$$

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